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Uniform Convergence of Cubic Spline Interpolants

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1. INTRODUCTION

Let C denote the space of all continuous functions f on [0, 1] which satisfy $f(0) = f(1)$. Let $\{\pi_n\}$ be a sequence of partitionings of [0, 1]. $\pi_n : 0 = x_0^{(0)} < \cdots < x_n^{(n)} = 1$. Let $\{L_n f\}$ be the sequence of *periodic cubic spline interpolants* associated with f and $\{\pi_n\}$, so that $L_n f(x_i) = f(x_i)$, $0 \leqslant i \leqslant n$, and $(L_n f)^{(j)}(0) = (L_n f)^{(j)}(1)$, $j = 0, 1, 2$.

A problem of some concern [4, 8] in the theory of spline approximation is the determination of simple necessary and sufficient conditions on $\{\pi_n\}$ to insure that $\{L_n f\}$ converges uniformly to f for all $f \in C$. Sharma and Meir [7] proved that

$$\text{if} \quad \beta_n \equiv \max_i h_i^{(n)} / \min_i h_i^{(n)} < \beta \qquad \text{for all} \quad n, \tag{1}$$

then $L_n f \to f$ uniformly for all $f \in C$. Here and following $h_i^{(n)} = x_i^{(n)} - x_{i-1}^{(n)}$, $\delta_n = \max_i h_i^{(n)}$, and for ease of notation the superscript n will be deleted if there is no danger of ambiguity. Nord [6] has shown the existence of $\{\pi_n\}$ (necessarily) with unbounded mesh ratio β_n and a continuous function f such that $L_n f \not\to f$ as $\delta_n \to 0$. That β_n need *not* be bounded for $L_n f \to f$ to hold was first observed by Cheney and Schurer [3]. Sharma and Meir [8] proved that

$$\text{if} \quad P_n \equiv \max_{|i-j|=1} h_i^{(n)} / h_j^{(n)} < \sqrt{2} \qquad \text{for all} \quad n, \tag{2}$$

then $L_n f \to f$ uniformly for all $f \in C$. This condition was later [4] relaxed by Cheney and Schurer to

$$P_n < 2 \qquad \text{for all } n \text{ sufficiently large.} \tag{3}$$

In [4], it was conjectured that (3) is also a necessary condition in order that $L_n f \to f$ for all $f \in C$. We observe that this conjecture is false since one can construct a $\{\pi_n\}$ satisfying (1) but not (3). Of course, (3) does not imply (1) either.

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The main purpose of this note is to establish a third sufficient condition, not implying (1) or (3), which insures that $L_n f \to f$ for all $f \in C$ (Theorem 1). In addition, the convergence of *nonperiodic cubic spline interpolants* to functions $f \in \mathbb{C}[0, 1]$ is established under similar conditions on $\{\pi_n\}$.

2. Convergence Theorem

Define (for a fixed π_n)

$$q_n(i) \equiv (h_i/(h_{i+1} + h_{i+2})) + (h_{i+1}/(h_{i-1} + h_i)), \qquad 1 \leqslant i \leqslant n, \tag{4}$$

where $h_0 \equiv h_n$, $h_{n+1} \equiv h_1$, and $h_{n+2} \equiv h_2$. Let

$$Q_n \equiv \max_i q_n(i). \tag{5}$$

We now state and prove the main result.

THEOREM 1. *If $P_n < P$ and $Q_n \leqslant Q < 2$ then for each $f \in C$*

$$\|f - L_n f\|_\infty \leqslant [3P^2 + 2(P+1)(2-Q)/2(2-Q)(P+1)]\, \omega(f; \delta_n), \tag{6}$$

where $\omega(f; \delta_n)$ is the modulus of continuity of f. Consequently, $L_n f \to f$ uniformly as $\delta_n \to 0$.

Remark. There exists $\{\pi_n\}$ satisfying the above conditions but such that $P_n > 2$ and β_n is unbounded. For example, consider $\{\pi_n\}$ such that for n even, $n = 2m$,

$$\begin{aligned}
h_m &= h_{m+1} = \delta_n \\
h_{m-1} &= h_{m+2} = 1/3\, \delta_n \\
h_{m-2} &= h_{m+3} = 1/3\, \delta_n \\
h_{m-3} &= h_{m+4} = 1/3^2\, \delta_n \\
h_{m-4} &= h_{m+5} = 1/3^2\, \delta_n \\
&\vdots \\
h_1 &= h_n = 1/3^k\, \delta_n, \qquad \text{where} \quad k = \begin{cases} (m-1)/2 & \text{if } m \text{ odd} \\ m/2 & \text{if } m \text{ even.} \end{cases}
\end{aligned} \tag{7}$$

For n odd, let $n + 1 = 2m$, ignore h_{n+1} in the last line of (7). For example, $n = 11$ yields $h_1 = \delta/27$, $h_2 = h_3 = h_{10} = h_{11} = \delta/9$, $h_4 = h_5 = h_8 = h_9 = \delta/3$, and $h_6 = h_7 = \delta = 27/103$. Clearly, $P_n = 3$ and β_n is unbounded as $n \to \infty$, but one can easily verify that the conditions in Theorem 1 are satisfied with $Q = 11/6$.

Proof of Theorem 1. Let $H^2([0, 1], \pi_n)$ be the subspace of $\mathbb{C}^1[0, 1]$ consisting of functions which are cubic polynomials in each subinterval of π_n, i.e., $H^2([0, 1], \pi_n)$ is the *smooth Hermite space* or order 2, [2]. Write

$$L_n f - f = (L_n f - V_n f) + (V_n f - f), \tag{8}$$

where $V_n f$ is the unique element of the smooth Hermite space $H^2([0, 1], \pi_n)$, such that $V_n f(x_i) = f(x_i)$ and $(V_n f)'(x_i) = 0$ for $0 \leqslant i \leqslant n$. The following two lemmas bound the terms on the right side of (8).

LEMMA 1. *$f \in C$ implies $\| V_n f - f \|_\infty \leqslant \omega(f; \delta_n)$.*

Proof. Using the notation of [5], for $x_{i-1} \leqslant x \leqslant x_i$

$$\begin{aligned} V_n f(x) &= f_{i-1} H_1(\bar{x}) + f_i H_2(\bar{x}) \\ &= f(x) + (f_{i-1} - f(x))\, H_1(\bar{x}) + (f_i - f(x))\, H_2(\bar{x}), \end{aligned}$$

since $(H_1 + H_2)(\bar{x}) = 1$ for all x. Lemma 1 follows immediately since $(| H_1 | + | H_2 |)(\bar{x}) = 1$ also.

LEMMA 2. *$f \in C$ implies $\| L_n f - V_n f \|_\infty \leqslant [3P^2/2(2 - Q)(P + 1)]\, \omega(f; \delta_n)$.*

Proof. Note that $L_n f$ and $V_n f$ are both elements of $H^2([0, 1], \pi_n)$ and as in [5, Eq. (11)] we have for $x_{i-1} \leqslant x \leqslant x_i$

$$\| L_n f - V_n f \|_\infty \leqslant K_i \, \| \{ | H_3(\bar{x}) | + | H_4(\bar{x}) | \} \|_\infty \leqslant h_i K_i / 4, \tag{9}$$

where $K_i = \max\{L_n f'(x_{i-1}), L_n f'(x_i)\}$.

Remark. Equation (11) in Ref. 5 contains a misprint. The term in braces should be written as in (9). The bound $\Delta/4$ given in [5] is for this latter quantity.

We now seek a bound on K_i for each i. Let $\mathbf{X}$ and $\mathbf{f}$ be $n \times 1$ vectors with

$$[\mathbf{X}]_i = (L_n f)'(x_i) \quad \text{and} \quad [\mathbf{f}]_i = 3\{(h_{i+1}/h_i)(f_i - f_{i-1}) + (h_i/h_{i+1})(f_{i+1} - f_i)\}.$$

Let

$$M = \begin{bmatrix} 2(h_1 + h_2) & h_1 & 0 \cdots 0 & h_2 \\ h_3 & 2(h_2 + h_3) & h_2 & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & \ddots & h_{n-1} \\ h_n & 0 \cdots 0 & h_1 & 2(h_n + h_1) \end{bmatrix},$$

then we must have from [1, Eq. (2.1.17)]

$$M\mathbf{X} = \mathbf{f}. \tag{10}$$

Define $D = \text{diag}\{2(h_1 + h_2), \ldots, 2(h_n + h_1)\}$. Then (10) is equivalent to

$$(MD^{-1})(D\mathbf{X}) = \mathbf{f}. \tag{10'}$$

The matrix $MD^{-1} = I + B$, where $\| B \|_\infty \leqslant Q/2 < 1$, and by [9, p. 61] it follows that $\|(I + B)^{-1}\|_\infty \leqslant 2/(2 - Q)$. Hence

$$\| D\mathbf{X} \|_\infty \leqslant 12P\omega(f; \delta_n)/(2 - Q).$$

Therefore, $|(L_n f)'(x_i)| \leqslant 6P\omega(f; \delta_n)/[(2 - Q)(h_i + h_{i+1})]$ and from (9), Lemma 2 follows immediately.[1] The bound in (6) then follows from the triangle inequality and the proof of Theorem 1 is complete.

The proof of Theorem 1 is easily modified to establish the following theorem.

THEOREM 2. *Let $S_n f$ be the unique cubic spline interpolant of a (continuous, not necessarily periodic) function $f \in \mathbb{C}[0, 1]$ such that $S_n f(x_i) = f(x_i)$, $0 \leqslant i \leqslant n$ and $(S_n f)'(0) = (S_n f)'(1) = 0$. If $P_n < P$ and $Q_n \leqslant Q < 2$ then*

$$\| f - S_n f \|_\infty \leqslant [3P^2 + 2(P + 1)(2 - Q)/2(2 - Q)(P + 1)]\, \omega(f; \delta_n).$$

Consequently, $S_n f \to f$ uniformly as $\delta_n \to 0$.

If the class $\mathbb{C}[0, 1]$ or C is restricted to $\text{Lip}_N 1 \equiv \{f \mid \omega(f, \delta) \leqslant N\delta\}$, then we have the following theorem.

THEOREM 3. *If $f \in \text{Lip}_N 1$ then independent of the mesh $\{\pi_n\}$*

$$\| S_n f - f \|_\infty \leqslant 5/2\, N\delta_n, \tag{11}$$

and if f is periodic, i.e., $f(0) = f(1)$ then

$$\| L_n f - f \|_\infty \leqslant 5/2\, N\delta_n. \tag{12}$$

Consequently, $S_n f \to f$ and $L_n f \to f$ uniformly as $\delta_n \to 0$.

Proof. The proof of Theorem 1 need only be modified as follows. From Eq. (10) we have

$$D^{-1}M\mathbf{X} = D^{-1}\mathbf{f},$$

and hence

$$\| X \|_\infty \leqslant \| (D^{-1}M)^{-1} \|_\infty \cdot \| D^{-1}f \|_\infty \leqslant 3 \left[\frac{\omega(f; h_i)}{h_i} + \frac{\omega(f; h_{i+1})}{h_{i+1}}\right] \leqslant 6N.$$

The remainder of the proof parallels the proof of Theorem 1 and is omitted.

[1] The author is indebted to the referee for the observation that $h_i/(h_i + h_{i+1}) \leqslant P/(P + 1)$ which improves the bound of 1 in the original manuscript.

Remark. If $\{\pi_n\}$ satisfies (2) or (3) then $\{\pi_n\}$ satisfies the hypothesis of Theorems 1 and 2. We need only note that $1/P_n \leqslant h_i/h_{i-1}$, $h_{i-1}/h_i \leqslant P_n$ and

$$\begin{aligned} q_n(i) &= h_i/h_{i+1}[1 + (h_{i+2}/h_{i+1})] + h_{i+1}/h_i[1 + (h_{i-1}/h_i)] \\ &< 1/(1 + 1/P_n)[(h_i/h_{i+1}) + (h_{i+1}/h_i)] \\ &< P_n/(1 + P_n)[1/P_n + P_n] = (P_n^2 + 1)/(P_n + 1), \end{aligned}$$

which is less than 2 for $P_n < 1 + \sqrt{2}$ and is maximized on $[1, P]$ at P. The author is indebted to Professor M. Marsden for this bound on $q_n(i)$ which improves the estimate $q_n(i) < P_n$ given in the original manuscript. From (6), with $Q = (P^2 + 1)/(P + 1)$, we have the following theorem.

THEOREM 4. *If $P_n \leqslant P < 1 + \sqrt{2}$ for all n, then for each $f \in C$,*

$$\|f - L_n f\|_\infty \leqslant (P^2 + 4P + 2)/2(-P^2 + 2P + 1)\, \omega(f; \delta_n).$$

This improves somewhat the bounds given in [4] or [8] for a given $P < 2$ and extends the admissible interval for P to $[1, 1 + \sqrt{2})$.

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